

SAW*-ALGEBRAS ARE ESSENTIALLY NON-FACTORIZABLE

SAEED GHASEMI

ABSTRACT. In this paper we solve a question of Simon Wassermann, whether the Calkin algebra can be written as tensor product of two infinite dimensional C*-algebras. More generally we show that there is no surjective *-homomorphism from a SAW*-algebra onto two infinite dimensional C*-algebras. We also discuss a conjecture about the structures of the *-homomorphisms from the tensorial powers of SAW*-algebras.

1. INTRODUCTION

It was shown by L. Ge [7], using free entropy, that if group von Neumann algebra of \mathbb{F}_2 , $L(\mathbb{F}_2)$, is written as von Neumann tensor product of two von Neumann algebras M and N then either M or N has to be \mathbb{M}_n for some n . In recent years, C*-tensor product theory and amenable actions of groups have provided some deep results related to the classification of von Neumann algebras. A C*-algebra \mathcal{A} is called *essentially non-factorizable* (it is also called *prime* for type II_1 -factors) if it can not be written as $\mathcal{B} \otimes_{\nu} \mathcal{C}$ where both \mathcal{B} and \mathcal{C} are infinite dimensional for some C*-algebra norm ν . It is shown by Simon Wassermann that the reduced C*-algebra of \mathbb{F}_2 , $C_r^*(\mathbb{F}_2)$, is essentially non-factorizable. In fact if $C_r^*(\mathbb{F}_2) = \mathcal{B} \otimes_{\nu} \mathcal{C}$, for some C*-norm ν and infinite dimensional C*-algebra \mathcal{B} then $\mathcal{C} = M_n$ with $n = 1$ and it was asked by him if the Calkin algebra is essentially non-factorizable. We prove that the answer to this question is positive.

Theorem 1.1. *Calkin algebra is essentially non-factorizable.*

It is well known that C*-algebras can be viewed as non-commutative topological spaces and the correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C*-algebras and unital *-homomorphisms. Each property of a locally compact Hausdorff space can be reformulated in terms of the function algebra $C_0(X)$, so it usually make sense to ask about these properties for noncommutative C*-algebras. SAW*-algebras were introduced by G.K. Pedersen [9] as non-commutative analogues of sub-Stonean spaces (also known as F-spaces) in topology, which are the locally compact Hausdorff spaces in which disjoint σ -compact open subspaces have disjoint compact closure. Analogously a C*-algebra \mathcal{A} is called an SAW*-algebra if for every two orthogonal elements x and y in \mathcal{A}_+ , there is an element e in \mathcal{A}_+ such that $ex = x$ and $ey = 0$.

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In [8] and [9] some of the properties of sub-Stonean spaces are generalized to SAW^* -algebras. It is proved (cf. [9]) that corona algebra of any σ -unital C^* -algebra is a SAW^* -algebra. In particular Calkin algebras are SAW^* -algebras. Another class of C^* -algebras, called countably degree-1 saturated C^* -algebras, are introduced in [5] by using model theoretic notions, where it has been shown that countably degree-1 saturated C^* -algebras as well as ultrapowers of C^* -algebras and relative commutant of these algebras (when they are separable) are SAW^* -algebras. In this paper we will use another property of sub-Stonean spaces to show that SAW^* -algebras are essentially non-factorizable.

In this paper we don't require any knowledge about sub-Stonean spaces and it's enough to know that $\beta\mathbb{N}$, the Čech- Stone compactification of \mathbb{N} , is a sub-Stonean space.

Let \mathcal{A} and \mathcal{B} be two unital C^* -algebras and $\mathbb{S}(\mathcal{A})$ and $\mathbb{P}(\mathcal{A})$ denote the space of all states and pure states on \mathcal{A} , respectively. For a $*$ -homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ consider the adjoint map $f^* : \mathbb{S}(\mathcal{B}) \longrightarrow \mathbb{S}(\mathcal{A})$ defined by $f^*(\phi) = \phi \circ f$. Recall that f^* is weak*-continuous and moreover it is injective if and only if f is surjective and it is surjective if and only if f is injective.

2. $*$ -HOMOMORPHISMS

We adopt standard notations from Ramsey theory and write $[\mathbb{N}]^2$ to denote the set of all $(m, n) \in \mathbb{N}^2$ such that $m < n$ and $\Delta^2\mathbb{N}$ to denote the diagonal of \mathbb{N}^2 . For spaces X and Y a rectangle is a subset of $X \times Y$ of the form $A \times B$ for $A \subset X$ and $B \subset Y$. We say a map f on $A \times B$ depends only on the first coordinate if $f(x, y) = f(x, z)$ for every (x, y) and (x, z) in $A \times B$. In [12] Van Douwen proved that for any continuous map $f : \beta\mathbb{N}^2 \rightarrow \beta\mathbb{N}$ there is a clopen $U \subset \beta\mathbb{N}$ such that $f \upharpoonright U^2$ depends on at most one coordinate and conjectured that there is a disjoint open cover of $\beta\mathbb{N}^2$ into such sets. In [3] I. Farah showed that for a sub-Stonean space Z , compact spaces X and Y , every continuous map $f : X \times Y \rightarrow Z$ is of a "very simple" form, which will be clear from Theorem 2.1 (in fact the theorem is proved for a larger class of spaces so called $\beta\mathbb{N}$ - spaces and product of any number of compact spaces). We sketch the proof of this theorem for the convenience of the reader.

Theorem 2.1 (I. Farah). *If f is a continuous map from $X \times Y$ into Z where X and Y are compact topological spaces and Z is a sub-Stonean space, then $X \times Y$ can be covered by finitely many clopen rectangles such that f depends on at most one coordinate on each of them.*

Proof. In [2], Theorem 3, it has been shown that for such f , either $X \times Y$ can be covered by finitely many rectangles such that f depends on at most one coordinate on each of them or there are sequences $x_i \in X$, $y_i \in Y$ such that for all i and all $j < k$ we have $f(x_i, y_i) \neq f(x_j, y_k)$. So we just need to show the second case does

not happen. Suppose $\{x_i\}$ and $\{y_i\}$ are sequences guaranteed by the second case. Define the map $g : \mathbb{N}^2 \rightarrow X \times Y$ by $g(m, n) = (x_m, y_n)$. g continuously extends to the map $\beta g : \beta\mathbb{N}^2 \rightarrow X \times Y$. Let $h : \beta\mathbb{N}^2 \rightarrow X \times Y$ be the continuous map defined by $h = f \circ \beta g$ and it has the property that $h(l, l) \neq h(m, n)$ for all l and all m, n such that $m < n$. But this contradicts the Corollary 7.6. in [3] which states that if $f : \beta\mathbb{N}^2 \rightarrow Z$ is a continuous map then the sets $f([\mathbb{N}]^2)$ and $f(\Delta^2\mathbb{N})$ have nonempty intersection. \square

As a corollary of this, if X and Y are infinite, any such f is not injective. For C^* -algebras, the product of non-commutative spaces corresponds to the tensor product of algebras. This motivates us to try to prove the theorem 1.1 using translation between notions in topology and commutative C^* -algebras.

Any such simple map between products of topological spaces is called "piecewise elementary" in [3]. We adopt this terminology for C^* -algebras and define the following:

Definition 2.2. For \mathcal{A}, \mathcal{B} and \mathcal{C} unital C^* -algebras, we say a unital $*$ -homomorphism $f : \mathcal{A} \rightarrow \mathcal{B} \otimes \mathcal{C}$ is piecewise elementary if there are finitely many projections p_1, \dots, p_s in \mathcal{B} and q_1, \dots, q_t projections in \mathcal{C} such that $\sum_{i=1}^s p_i = 1_{\mathcal{B}}$ and $\sum_{j=1}^t q_j = 1_{\mathcal{C}}$ and for every $1 \leq i \leq s$ and $1 \leq j \leq t$ the projection $(p_i \otimes q_j)$ commutes with the image of f and for $a \in \mathcal{A}$ either there is a $*$ -homomorphism $g : \mathcal{A} \rightarrow \mathcal{B}$ such that $(p_i \otimes q_j)f(a)(p_i \otimes q_j) = (p_i g(a) p_i) \otimes q_j$ or there is a $*$ -homomorphism $h : \mathcal{A} \rightarrow \mathcal{C}$ such that $(p_i \otimes q_j)f(a)(p_i \otimes q_j) = p_i \otimes (q_j h(a) q_j)$.

By Gelfand-Naimark transformation we can restate Farah's theorem for commutative C^* -algebras.

Theorem 2.3. Any unital $*$ -homomorphism $f : \mathcal{A} \rightarrow \mathcal{B} \otimes \mathcal{C}$, where \mathcal{A}, \mathcal{B} and \mathcal{C} unital commutative C^* -algebras and \mathcal{A} is a SAW*-algebra, is piecewise elementary.

Note that in particular every element in the image of f is finite sum of elementary tensor products and if \mathcal{A} is a commutative SAW*-algebra with no projections (e.g. $\mathcal{A} = C(X)$ where X is a connected sub-Stonean space like $\beta\mathbb{R} \setminus \mathbb{R}$) the image of f can be identified with a subspace of \mathcal{B} or \mathcal{C} .

Lemma 2.4. If \mathcal{B} is an infinite-dimensional, unital C^* -algebra, we can find nonzero orthogonal sequence $0 \leq a_n \leq 1$ of elements of \mathcal{B} and a sequence of states on \mathcal{B} , $\{\phi_n\}$, such that $\phi_n(a_n) = 1$ and $\phi_n(a_m) = 0$ if $m \neq n$.

Proof. It is well-known that (refer to [6]) any maximal abelian subalgebra (MASA) of an infinite-dimensional C^* -algebra \mathcal{B} is also infinite-dimensional. If not then there are orthogonal projections with finite rank $\{p_1, p_2, \dots, p_n\}$ in the MASA such that $\sum_{i=1}^n p_i = 1$. So $\mathcal{B} = \sum_{i,j=1}^n p_i \mathcal{B} p_j$ which means \mathcal{B} is finite-dimensional. By Gelfand-Naimark theorem identify this subalgebra with $C(X)$, where X is a compact Hausdorff space. We can also identify (see for example [4]) the set of pure states of $C(X)$ with X . Since X is infinite normal space we can choose a discrete sequence of pure states $\{\phi_n\}$ in X and find pairwise disjoint sequence $\{U_n\}$ of open neighborhoods of $\{\phi_n\}$. By Uryson's lemma we get an orthogonal sequence $0 \leq a_n \leq 1$ in $C(X)$ such

that $\phi_n(a_n) = a_n(\phi_n) = 1$ and a_n vanishes outside of U_n . So $\phi_n(a_m) = a_m(\phi_n) = 0$ if $m \neq n$. Now by Hahn-Banach extension theorem extend ϕ_n to a functional on \mathcal{B} of norm 1. Since $\phi_n(I) = 1$, this extension is a state. \square

Note that if ϕ is a state on a C^* -algebra \mathcal{A} and $\phi(a) = 1$ for $0 \leq a \leq I$, as a consequence of Cauchy-Schwartz inequality for states we have $\phi(b) = \phi(aba)$ for any $b \in \mathcal{A}$ (c.f. [4]).

Lemma 2.5. *For every sequence of states $\{\phi_n\}$ on a SAW^* -algebra \mathcal{A} if there exists sequence of orthogonal elements of norm one $0 \leq a_n \leq 1$ in \mathcal{A} such that $\phi_n(a_n) = 1$ and $\phi_n(a_m) = 0$ if $m \neq n$ then the weak*-closure of $\{\phi_n\}$ is homeomorphic to $\beta\mathbb{N}$.*

Proof. Let D be a subset of \mathbb{N} . We show that $\overline{\{\phi_n : n \in D\}} \cap \overline{\{\phi_n : n \in D^c\}} = \emptyset$. Take $\psi \in \overline{\{\phi_n : n \in D\}}$. Since \mathcal{A} is a SAW^* -algebra we can find a positive element e such that $ea_n = a_n$ for every $n \in D$ and $ea_n = 0$ for every $n \in D^c$. We claim that $\psi(e) = 1$. Suppose not and let $\psi(e) = \alpha < 1$. Note that if $n \in D$, we have $\phi_n(e) = \phi_n(ea_n) = \phi_n(a_n) = 1$ and if $n \in D^c$ we have $\phi_n(e) = \phi_n(ea_n) = \phi_n(0) = 0$. Let

$$U = \{\phi \in \mathbb{S}(\mathcal{A}) : |\phi(e) - \psi(e)| < (1 - \alpha)/2\}$$

U is an open neighborhood of ψ . If $n \in D$ we have

$$|\phi_n(e) - \psi(e)| = 1 - \alpha$$

which implies that $U \cap \{\phi_n : n \in D\} = \emptyset$. Therefore ψ is not in $\overline{\{\phi_n : n \in D\}}$. So $\psi(e) = 1$.

Let

$$V = \{\phi \in \mathbb{S}(\mathcal{A}) : |\phi(e) - \psi(e)| < 1\}$$

Since for any $n \in D^c$ we have $|\phi_n(e) - \psi(e)| = 1$, so ϕ_n is not in V . Therefore ψ is not in $\overline{\{\phi_n : n \in D^c\}} = \emptyset$.

Now let $F : \beta\mathbb{N} \rightarrow \overline{\{\phi_n : n \in \mathbb{N}\}}$ be the continuous map such that $F(n) = \phi_n$. Let \mathcal{U} and \mathcal{V} be two distinct ultrafilters in $\beta\mathbb{N}$ and pick $X \subseteq \mathbb{N}$ such that $X \in \mathcal{U}$ but X is not in \mathcal{V} .

$$F(\mathcal{U}) \in \overline{F(\overline{X})} \subseteq \overline{F(X)} = \overline{\{\phi_n : n \in X\}}$$

Similarly $F(\mathcal{V}) \in \overline{\{\phi_n : n \in X^c\}}$. So F is injective and clearly surjective, since $\beta\mathbb{N}$ is compact and $\overline{\{\phi_n\}}$ is Hausdorff, therefore F is a homeomorphism. \square

Theorem 2.6. *Any SAW^* algebra is essentially non-factorizable.*

Proof. suppose \mathcal{B} and \mathcal{C} be two infinite dimensional unital C^* -algebras. We would use the minimal tensor product of C^* -algebras but the result would remain unchanged for any tensor product of C^* -algebras, since the identity map from the algebraic tensor product of \mathcal{B} and \mathcal{C} equipped with any C^* -norm ν into $\mathcal{B} \otimes_{\min} \mathcal{C}$ is norm decreasing and can be extended to a surjective $*$ -homomorphism from $\mathcal{B} \otimes_{\nu} \mathcal{C}$ onto $\mathcal{B} \otimes_{\min} \mathcal{C}$.

Let $f : \mathcal{A} \rightarrow \mathcal{B} \otimes_{\min} \mathcal{C}$ be a surjective $*$ -homomorphism and $f^* : \mathbb{S}(\mathcal{B} \otimes_{\min} \mathcal{C}) \rightarrow \mathbb{S}(\mathcal{A})$ be the injective and continuous map defined by $f^*(\gamma) = \gamma \circ f$. Suppose $\{\phi_n\}$ and $\{\psi_n\}$ be the sequences of states and $\{b_n\}$ and $\{c_n\}$ be the orthogonal sequences

of positive elements in \mathcal{B} and \mathcal{C} respectively, guaranteed by lemma 2.4. Now let $\gamma_{m,n} = f^*(\phi_m \otimes \psi_n)$ and choose $a_{m,n} \in f^{-1}(b_m \otimes c_n)$. It is clear that the $\gamma_{m,n}$ are states on \mathcal{A} and $\{\gamma_{m,n}\}$ is an orthogonal sequence of positive elements of \mathcal{A} .

Claim: $\overline{\{\gamma_{m,n}\}}$ is homeomorphic to $\beta\mathbb{N}$.

We have:

$$\begin{aligned} \gamma_{m,n}(a_{m,n}) &= f^*(\phi_m \otimes \psi_n)(f^{-1}(b_m \otimes c_n)) \\ &= ((\phi_m \otimes \psi_n) \circ f)(f^{-1}(b_m \otimes c_n)) \\ &= (\phi_m \otimes \psi_n)(b_m \otimes c_n) \\ &= \phi_m(b_m) \psi_n(c_n) \\ &= 1. \end{aligned}$$

Similar calculation shows if $(m, n) \neq (m', n')$ then $\gamma_{m,n}(a_{m',n'}) = 0$.

For each $\gamma_{m,n}$, let $U_{m,n}$ open neighborhood around $\gamma_{m,n}$ defined by:

$$\begin{aligned} U_{m,n} &= \{\psi \in \mathbb{S}(\mathcal{A}) : |\psi(a_{m,n}) - \gamma_{m,n}(a_{m,n})| < \epsilon\} \\ (1) \quad &= \{\psi \in \mathbb{S}(\mathcal{A}) : \psi(a_{m,n}) > 1 - \epsilon\} \end{aligned}$$

For $\epsilon < 1$. Now if $(i, j) \neq (m, n)$, $\gamma_{i,j}$ is not in $U_{m,n}$. Hence $\{\gamma_{m,n}\}$ is a discrete subset of $\mathbb{S}(\mathcal{A})$. Using lemma 2.5 we have $\overline{\{\gamma_{m,n}\}}$ is homeomorphic to $\beta\mathbb{N}$. Now let $A = f^{*-1}(\overline{\{\gamma_{m,n} : (m, n) \in \mathbb{N}^2\}})$. A is a compact subset of $\mathbb{S}(\mathcal{B} \otimes_{\min} \mathcal{C})$ containing $\{(\phi_m \otimes \psi_n) : (m, n) \in \mathbb{N}^2\}$ and by the claim above $\overline{\{\gamma_{m,n} : (m, n) \in \mathbb{N}^2\}}$ is homeomorphic to $\beta\mathbb{N}$. Now by applying the lemma 2.1 to the function $f^* \upharpoonright A$, we get a contradiction since f^* is injective. \square

3. CONCLUDING REMARK

The analogues of the theorem 2.3 for non-commutative C^* -algebras is not true since being piecewise elementary implies that every element in the image of the $*$ -homomorphism is finite sum of elementary tensor products (e.g. the isomorphism from $B(H)$ onto $M_2 \otimes B(H)$ is not piecewise elementary). But one might hope for a similar but weaker theorem to be true for non-commutative C^* -algebras. This would provide us a strong tool to study the automorphisms between tensorial powers of Calkin algebra.

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DEPARTMENT OF MATHEMATICS, YORK UNIVERSITY, TORONTO, CA